Consider a random sample of length n for one individual $i = 1$ with response $\mathbf{Y}_i = (Y_1, ..., Y_n)^\top$ and co-variate design matrix

$$\mathbf{X}_i = \begin{pmatrix} 1 & x_{i1} \\ \vdots & \vdots \\ 1 & x_{in} \end{pmatrix}$$

with $x_{ij} \in \mathbb{R},\ j = 1, ..., n$. We consider the linear model

$$\mathbf{Y}_i = \mathbf{X}_i \boldsymbol{\beta} + \boldsymbol{\xi}_i$$

with coefficient vector $\boldsymbol{\beta} \in \mathbb{R}^2$ and $\boldsymbol{\xi}_i = (\xi_{i1}, ..., \xi_{in})^\top$ a random vector of length n with co-variance matrix V_i.

Note: For simplicity we assume data was only sampled for a single individual. Nevertheless, the index 'i' for the individuals is introduced in preparation for the lecture.

b) Compute $\text{Cov}(\hat{\boldsymbol{\beta}})$ for the simple least squares estimator $\hat{\boldsymbol{\beta}}$ for $\boldsymbol{\beta}$.

- The least square estimate is given by

$$\hat{\beta} = (X_i^T X_i)^{-1} X_i^T Y_i$$

- $$\text{Cov}(Y_i) = \text{Cov}(X_i \beta + \xi_i) = \text{Cov}(\xi_i) = V_i$$

these are constant

$$\Rightarrow \text{Cov}(\hat{\beta}) = \text{Cov}(\underbrace{(X_i^T X_i)^{-1} X_i^T}_{A} Y_i)$$

$$\overset{a)}{=} (X_i^T X_i)^{-1} X_i^T \, \text{Cov}(Y_i) \left((X_i^T X_i)^{-1} X_i^T\right)^T$$

$$= (X_i^T X_i)^{-1} X_i^T V_i X_i (X_i^T X_i)^{-1}$$

This is only well defined if $(X_i^T X_i)$ is invertible

symmetric