Lösung Aufgabe 1a)

Allgemeine Form der multivariaten Normalverteilung:

$$\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

$$f(x_1, \ldots, x_k) = \frac{1}{\sqrt{(2\pi)^k |\boldsymbol{\Sigma}|}} \cdot \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right),$$

mit $|\boldsymbol{\Sigma}|$ der Determinante von $\boldsymbol{\Sigma}$.

<u>Marginale Likelihood:</u>

$$\mathbf{Y}_i \sim \mathcal{N}(\mathbf{X}_i\boldsymbol{\beta}, \mathbf{V}_i(\boldsymbol{\alpha}))$$

$$L_{ML}(\boldsymbol{\beta}, \boldsymbol{\alpha}) = \prod_{i=1}^{N} \left\{ (2\pi)^{-\frac{n_i}{2}} \cdot |\mathbf{V}_i(\boldsymbol{\alpha})|^{-\frac{1}{2}} \cdot \exp\left(-\frac{1}{2}(\mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta})^T \mathbf{V}_i(\boldsymbol{\alpha})^{-1}(\mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta})\right)\right\}$$

$$\ell_{ML}(\boldsymbol{\beta}, \boldsymbol{\alpha}) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{N}\log(|\mathbf{V}_i(\boldsymbol{\alpha})|) - \frac{1}{2}\left\{\sum_{i=1}^{N}(\mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta})^T \mathbf{V}_i(\boldsymbol{\alpha})^{-1}(\mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta})\right\}$$

<u>Konditionale Likelihood:</u>

$$\mathbf{Y}_i|\mathbf{b}_i \sim \mathcal{N}(\mathbf{X}_i\boldsymbol{\beta} + \mathbf{Z}_i\mathbf{b}_i, \boldsymbol{\Sigma}_i)$$

$$L_{ML}(\boldsymbol{\beta}, \boldsymbol{b}_i) = \prod_{i=1}^{N} \left\{ (2\pi)^{-\frac{n_i}{2}} \cdot |\boldsymbol{\Sigma}_i|^{-\frac{1}{2}} \cdot \exp\left(-\frac{1}{2}(\mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta} - \mathbf{Z}_i\mathbf{b}_i)^T \boldsymbol{\Sigma}_i^{-1}(\mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta} - \mathbf{Z}_i\mathbf{b}_i)\right)\right\}$$

$$\ell_{ML}(\boldsymbol{\beta}, \boldsymbol{b}_i) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{N}\log(|\boldsymbol{\Sigma}_i|) - \frac{1}{2}\left\{\sum_{i=1}^{N}(\mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta} - \mathbf{Z}_i\mathbf{b}_i)^T \boldsymbol{\Sigma}_i^{-1}(\mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta} - \mathbf{Z}_i\mathbf{b}_i)\right\}$$